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# A note on the Picard motive of a variety

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## Abstract

We prove that the Picard motive of a smooth projective variety and the Picard motive of its Albanese variety are isomorphic, under the assumption that both the variety and its Albanese variety have dimension at least 2.

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## 1. Introduction

For various Weil cohomology theories ( $H^*(\cdot)$ , e.g., singular cohomology, étale cohomology, etc.), we have the well-known isomorphism

$$H^1(X) \simeq H^1(J_X),$$

where  $X$  is a smooth projective variety and  $J_X$  is its Albanese variety, both defined over some field  $k$ . Therefore, it is natural to ask if a similar isomorphism holds in the category of pure Chow motives ( $\mathcal{M}$ ). More precisely, if we let  $h^1(X)$  and  $h^1(J_X)$  denote the Picard motives of  $X$  and  $J_X$ , respectively [1,2], is there an isomorphism

$$h^1(X) \stackrel{?}{\simeq} h^1(J_X) \tag{1}$$

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In this note, we prove that, as expected, we have such an isomorphism, under the assumption that both the variety  $X$  and its Albanese variety  $J_X$  have dimension at least 2. In fact, this isomorphism is provided by the isomorphism of the Picard varieties of  $X$  and  $J_X$ .

## 2. Notation and preliminaries

The Picard motive was first introduced by Murre in [1]. Later, Scholl gave an excellent exposition of the theory in [2]. Here, we will follow Scholl's notation with minor exceptions. We write  $P_X$  for the Picard variety of  $X$ , and  $J_X$  (or  $A$ ) for its Albanese variety (i.e., the dual of  $P_X$ ). We denote by  $d_X$  the dimension of  $X$  and by  $d_A$  the dimension  $J_X$ . They are both assumed to be greater than 1. For a morphism of varieties  $\phi : X \rightarrow Y$ , we denote by  $J_\phi$  (respectively,  $P_\phi$ ) the induced morphism  $J_X \rightarrow J_Y$  (respectively,  $P_Y \rightarrow P_X$ );  $[\Gamma_\phi]$  denotes the (rational) equivalence class of the graph of  $\phi$ , and  $[\Gamma_\phi]^t$  denotes the class of the transpose of its graph. If we let  $\sigma : X \rightarrow J_X$  denote the Albanese map (after fixing a point  $x_0 \in X$  and  $\sigma(x_0) = e$ , the identity element with respect to the group operation on  $J_X$ ), we have  $J_\sigma = \text{id}_{J_X}$ , and  $P_\sigma : P_{J_X} \xrightarrow{\sim} P_X$  is an isomorphism.

For the sake of completeness, we include some results of Scholl here.

**Theorem 1** ([2, Theorem 3.9]). *Let  $X$  and  $Y$  be smooth, projective varieties (defined over some field  $k$ ) of dimensions  $d_X$  and  $d_Y$ , respectively.*

1. *Then there is an isomorphism*

$$\text{Hom}_{\text{Ab}}(J_X, P_Y) \otimes \mathbb{Q} \xrightarrow{\sim} \frac{A^1(X \times Y)}{p_1^* A^1(X) + p_2^* A^1(Y)},$$

where  $X \xleftarrow{p_1} X \times Y \xrightarrow{p_2} Y$  are projections and  $A^1(\cdot)$  denotes the Chow group of codimension 1 cycles with rational coefficients.

2. *Let  $\zeta \in A^{d_X}(X)$  and  $\eta \in A^{d_Y}(Y)$  be fixed zero cycles of positive degree. Then there is an isomorphism*

$$\text{Hom}_{\text{Ab}}(J_X, P_Y) \otimes \mathbb{Q} \xrightarrow{\Omega} \{c \in A^1(X \times Y) \mid c \circ \zeta_* = 0 = \eta^* \circ c\}.$$

(For definitions of  $\zeta_*$  and  $\eta^*$ , see [2, p.169].)

**Proposition 1** ([2, Proposition 3.10]). *Let  $X, Y, X'$  and  $Y'$  be smooth, projective varieties and  $\phi : X' \rightarrow X, \psi : Y' \rightarrow Y$  and  $\mu : J_X \rightarrow P_Y$  be morphisms. Then*

$$\Omega(P_\psi \circ \mu) = [\Gamma_\psi]^t \circ \Omega(\mu) \quad \text{and} \quad \Omega(\mu \circ J_\phi) = \Omega(\mu) \circ [\Gamma_\phi],$$

where  $\Omega$  is the isomorphism in the [Theorem 1](#).

Let  $C_X$  and  $C_A$  be one-dimensional linear sections of  $X$  and  $J_X$ , respectively. Then

$$\xi_X := [\Gamma_{i_X}] \circ [\Gamma_{i_X}]^t \quad \text{and} \quad \xi_A := [\Gamma_{i_A}] \circ [\Gamma_{i_A}]^t,$$

where  $i_X : C_X \hookrightarrow X$  and  $i_A : C_A \hookrightarrow J_X$  denote the embedding maps. We have the induced isogenies

$$\begin{aligned} \alpha_X : P_X &\xrightarrow{P_{i_X}} P_{C_X} = J_{C_X} \xrightarrow{J_{i_X}} J_X \\ \alpha_A : P_{J_X} &\xrightarrow{P_{i_A}} P_{C_A} = J_{C_A} \xrightarrow{J_{i_A}} J_{J_X} = J_X. \end{aligned}$$

We will denote the inverse isogenies by  $\beta_X$  and  $\beta_A$ , respectively. By taking a suitable common multiple if necessary, we can write

$$\alpha_X \circ \beta_X = n \cdot id_{J_X} = \alpha_A \circ \beta_A.$$

Then, the correspondences

$$p_1^X := \frac{1}{n} \Omega(\beta_X) \circ \xi_X, \quad p_{2d_X-1}^X := (p_1^X)^t, \quad \pi_1^X := p_1^X \circ \left(1 - \frac{1}{2} p_{2d_X-1}^X\right)$$

are projectors, and the Picard motive is given by

$$h^1(X) := (X, \pi_1^X).$$

In fact,  $\pi_1^X = p_1^X$  when  $d_X \geq 3$  [2].

### 3. The isomorphism

We would like to prove that  $h^1(X)$  and  $h^1(J_X)$  are isomorphic in  $\mathcal{M}$ . In fact, it is enough to prove the isomorphism of  $(X, p_1^X)$  and  $(J_X, p_1^A)$ , as  $\pi_1^X \circ p_1^X : (X, p_1^X) \rightarrow h^1(X)$  and  $p_1^X \circ \pi_1^X : h^1(X) \rightarrow (X, p_1^X)$  are mutually inverse isomorphisms (of course, the same holds for  $p_1^A$  and  $\pi_1^A$ ). We define

$$\theta_1 := \frac{1}{n} p_1^A \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ \xi_X \circ p_1^X \in \text{Hom}_{\mathcal{M}}((X, p_1^X), (J_X, p_1^A))$$

$$\theta_2 := \frac{1}{n} p_1^X \circ \Omega(\beta_X) \circ \xi_A \circ p_1^A \in \text{Hom}_{\mathcal{M}}((J_X, p_1^A), (X, p_1^X)).$$

Note that, in the definition of  $\theta_2$ , we interpret  $\beta_X$  as a morphism  $J_{J_X} \rightarrow P_X$ , namely  $\beta_X = \beta_X \circ J_\sigma^{-1}$  but  $J_\sigma = id_{J_X}$ .

We will show that  $\theta_1$  and  $\theta_2$  are mutually inverse isomorphisms.

$$\begin{aligned} \theta_2 \circ \theta_1 &= \frac{1}{n^2} p_1^X \circ \Omega(\beta_X) \circ \xi_A \circ p_1^A \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ \xi_X \circ p_1^X \\ &= \frac{1}{n^3} p_1^X \circ \Omega(\beta_X) \circ [\Gamma_{i_A}] \circ [\Gamma_{i_A}]^t \circ \Omega(\beta_A) \circ [\Gamma_{i_A}] \circ [\Gamma_{i_A}]^t \\ &\quad \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ \xi_X \circ p_1^X \\ &= \frac{1}{n^3} p_1^X \circ \Omega(\beta_X \circ J_{i_A}) \circ \Omega(P_{i_A} \circ \beta_A \circ J_{i_A}) \circ \Omega(P_{i_A} \circ \beta_A \circ J_\sigma) \circ \xi_X \circ p_1^X \\ &= \frac{1}{n^3} p_1^X \circ \Omega(\beta_X \circ J_{i_A} \circ P_{i_A} \circ \beta_A \circ J_{i_A} \circ P_{i_A} \circ \beta_A \circ id_{J_X}) \circ \xi_X \circ p_1^X \\ &= \frac{1}{n^3} p_1^X \circ \Omega(\beta_X \circ \alpha_A \circ \beta_A \circ \alpha_A \circ \beta_A) \circ \xi_X \circ p_1^X \\ &= p_1^X \circ p_1^X \circ p_1^X = p_1^X. \end{aligned}$$

For the converse, we have

$$\begin{aligned} \theta_1 \circ \theta_2 &= \frac{1}{n^2} p_1^A \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ \xi_X \circ p_1^X \circ \Omega(\beta_X) \circ \xi_A \circ p_1^A \\ &= \frac{1}{n^3} p_1^A \circ \Omega(\beta_A) \circ [\Gamma_\sigma] \circ [\Gamma_{i_X}] \circ [\Gamma_{i_X}]^t \end{aligned}$$

$$\begin{aligned}
& \circ \Omega(\beta_X) \circ [\Gamma_{i_X}] \circ [\Gamma_{i_X}]^t \circ \Omega(\beta_X) \circ \xi_A \circ p_1^A \\
&= \frac{1}{n^3} p_1^A \circ \Omega(\beta_A \circ J_\sigma \circ J_{i_X}) \circ \Omega(P_{i_X} \circ \beta_X \circ J_{i_X}) \circ \Omega(P_{i_X} \circ \beta_X) \circ \xi_A \circ p_1^A \\
&= \frac{1}{n^3} p_1^A \circ \Omega(\beta_A \circ id_{J_X} \circ \alpha_X \circ \beta_X \circ \alpha_X \circ \beta_X) \circ \xi_A \circ p_1^A \\
&= p_1^A \circ p_1^A \circ p_1^A = p_1^A.
\end{aligned}$$

Hence, the isomorphism (1) holds.

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